Govt. Engineering College Ajmer

III Sem.(CE) Main (2017– 18)

MM: 10 Sub:Advanced Engg. Mathematics (II Mid–Term Test)

Time: 1Hr.

Note: Each question carries equal marks.

1. Find the Z-transform and ROC of $u_n = \begin{cases} 4^n & n < 0 \\ 3^n & n \geq 0 \end{cases}$

2. Find the inverse Z-transform of $\dfrac{1}{(z-a)^2}$ when

   (i)|z|<a (ii)|z|>a

3. Find $F_c\left(\dfrac{1}{1+x^2}\right)$

4. Prove that $\displaystyle\int_0^\infty \frac{\cos sx}{1+s^2}\,ds = \frac{\pi}{2}e^{-x}, \; x \geq 0$

Solutions

(CE-III Sem 2017-18
SUB - AEM
II Mid term test)

① $u_n = \begin{cases} 4^n & n<0 \\ 3^n & n \geq 0 \end{cases}$

$$Z(u_n) = \sum_{n=-\infty}^{\infty} u_n z^{-n}$$

$$= \sum_{n=-\infty}^{-1} 4^n z^{-n} + \sum_{n=0}^{\infty} 3^n z^{-n}$$

$$= \left(- - - + \frac{z^2}{4^2} + \frac{z}{4}\right) + \left(1 + \frac{3}{z} + \frac{3^2}{z^2} + - - -\right.$$

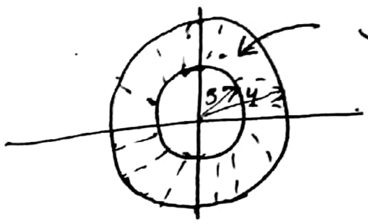
If $|z|<4$ $\qquad$ If $|z|>3$

$$= \frac{z/4}{1-(z/4)} + \frac{1}{1-(3/z)}$$

$$= \frac{z}{4-z} + \frac{z}{z-3}$$

$$= \frac{z}{(z-3)(4-z)} \quad \text{if } 3<|z|<4$$

ie ROC



$3<|z|<4$
excluding boundary.

② $z^{-1}\left(\frac{1}{(z-a)^2}\right)$

(i) $|z|<a$

$\Rightarrow \left|\frac{z}{a}\right|<1$

ie $\bar{u}(z) = \frac{1}{(z-a)^2} = \frac{1}{a^2}\left(1-\frac{z}{a}\right)^{-2}$

$$= \frac{1}{a^2}\left[1 + \frac{2z}{a} + \frac{(-2)(-3)}{2!}\left(\frac{-z}{a}\right)^2 + \cdots\right.$$

$$= \frac{1}{a^2}\left(1 + \frac{2z}{a} + \frac{3z^2}{a^2} + \cdots\right)$$

$$\Rightarrow u_n = (1-n)\, a^{n-2} \quad \text{for } n \leq 0$$

(ii) If $|z| > a$

$$\left|\frac{a}{z}\right| < 1$$

$$\bar{u}(z) = \frac{1}{z^2\left(1-\frac{a}{z}\right)^2} = \frac{1}{z^2}\left(1-\frac{a}{z}\right)^{-2}$$

$$= \frac{1}{z^2} + 2a\left(\frac{1}{z}\right)^3 + (3a^2)\left(\frac{1}{z}\right)^4 + \cdots$$

ie $u_n = (n-1)\, a^{n-2}$ for $n \geq 2$

(3) $I = F_c\left(\frac{1}{1+x^2}\right)$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{1}{1+x^2}\right) \cos sx \, dx$$

Differentiating w.r.t s

$$\frac{dI}{ds} = +\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{-x}{1+x^2} \sin sx \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(x^2+1-1)}{x(1+x^2)} \sin sx \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \left[\int_0^\infty \frac{\sin sx}{x} dx - \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx\right]$$

$$= -\sqrt{\frac{2}{\pi}} \left[\underbrace{\frac{\pi}{2}}_{\text{(using Laplace transform)}} - \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx\right] . \quad \text{--- (1)}$$

question ...

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Again differentiating w.r.t s.

$$\frac{d^2 I}{ds^2} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{x \cos sx}{x(1+x^2)} dx$$

$$= I$$

$$\Rightarrow \frac{d^2 I}{ds^2} - I = 0$$

$$\Rightarrow I = c_1 e^{s} + c_2 e^{-s} \quad \Rightarrow \text{put } s=0 \quad I = c_1 + c_2$$

$$\frac{dI}{ds} = c_1 e^{s} - c_2 e^{-s}$$

Also $I = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{dx}{1+x^2}$

$$= \sqrt{\frac{\pi}{2}}$$

ie $c_1 + c_2 = \sqrt{\frac{\pi}{2}}$ — (2)

Put $s=0$ in $\frac{dI}{ds}$ & in (1)

$$\left.\begin{aligned} \frac{dI}{ds} &= c_1 - c_2 \\ \frac{dI}{ds} &= -\sqrt{\frac{\pi}{2}} \end{aligned}\right\} \text{ ie } c_1 - c_2 = -\sqrt{\frac{\pi}{2}} \quad \text{— (3)}$$

From (2) & (3) $c_1 = 0$, $c_2 = \sqrt{\frac{\pi}{2}}$

ie $I = \sqrt{\frac{\pi}{2}}\, e^{-s}$.

(4) Let $I = \int_0^{\infty} \frac{\cos sx}{1+s^2} ds$

Diff w.r.t x

$$\frac{dI}{dx} = -\int_0^{\infty} \frac{(s^2+1-1)\sin sx\, ds}{s(1+s^2)}$$

Continuing in the same manner as in Question (3) we get the result.