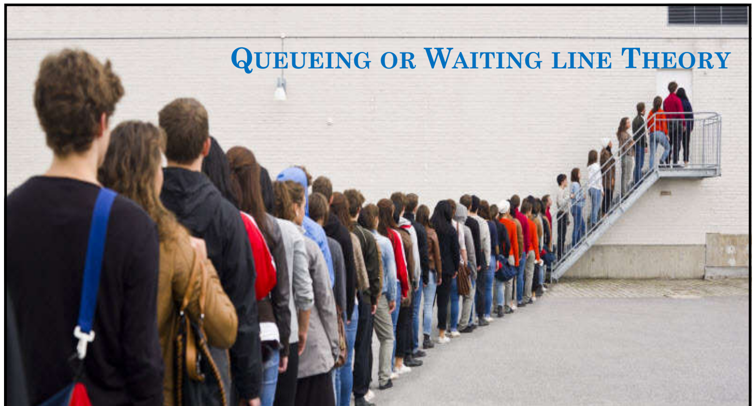

Queueing or Waiting line Theory

Examples

- Commercial Queuing Systems
  - Commercial organizations serving external customers
  - Ex. Dentist, bank, ATM, gas stations, plumber, garage …
- Transportation service systems
  - Vehicles are customers or servers
  - Ex. Vehicles waiting at toll stations and traffic lights, trucks or ships waiting to be loaded, taxi cabs, fire engines, elevators, buses …
- Business-internal service systems
  - Customers receiving service are internal to the organization providing the service
  - Ex. Inspection stations, conveyor belts, computer support …
- Social service systems
  - Ex. Judicial process, the ER at a hospital, waiting lists for organ transplants or student dorm rooms …

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Queueing theory

- Queues arise when the short term demand for service exceeds the capacity.
  - Most often caused by random variation in service times and the times between customer arrivals.
- Mathematical analysis of queues and waiting times in stochastic systems.
  - Used extensively to analyze production and service processes exhibiting random variability in market demand (arrival times) and service times.
- Helps managers to better understand systems in manufacturing, service, and maintenance.
- Provides competitive advantage and cost saving.

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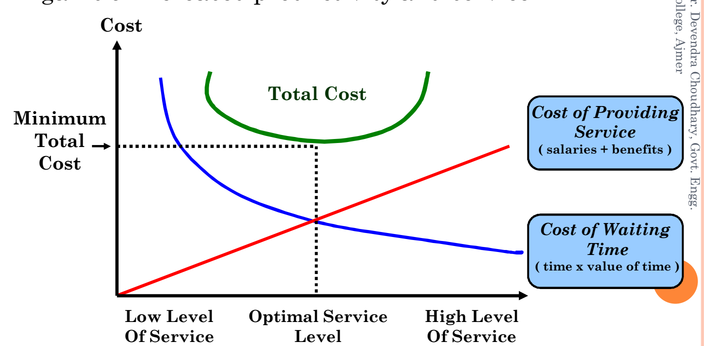
Why is Queuing Analysis Important?

- Queuing and waiting time analysis is particularly important in service systems
  - Large costs of waiting and of lost sales due to waiting
- Need to balance the cost of increased capacity against the gains of increased productivity and service



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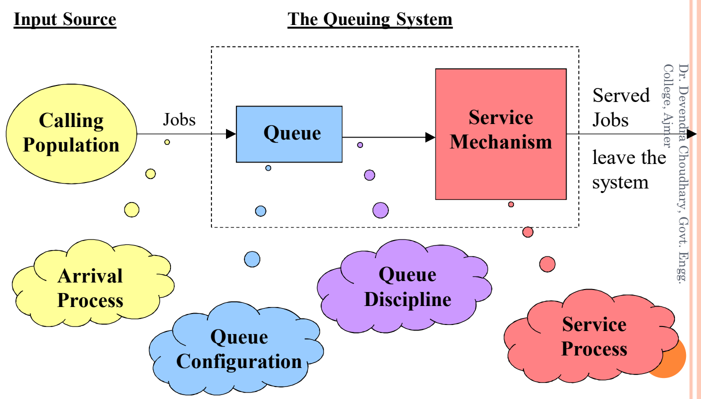
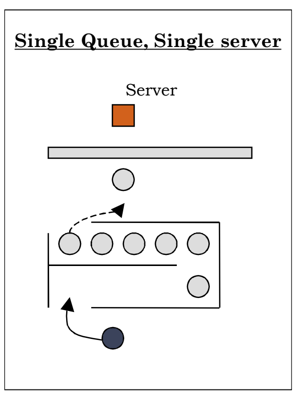
Components of a Basic Queuing Process



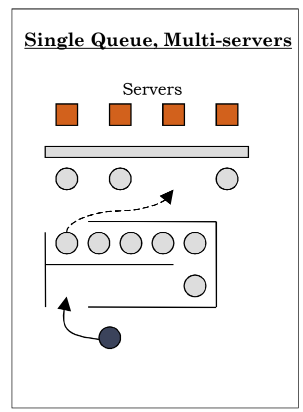
Specification of a Queue

- Source
  - Finite
  - Infinite
- Arrival Process
- Service Time Distribution
- Maximum Queueing System Capacity
- Number of Servers
- Queue Discipline
  - Most commonly used principle is FIFO.
  - Other rules are, for example, LIFO, SIRO etc.

Queue Configurations

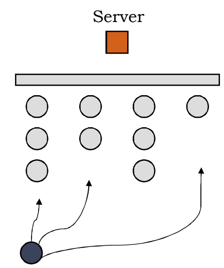


Queue Configurations



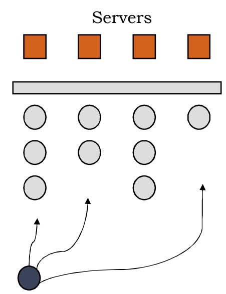
Queue Configurations

Multiple Queues, Single Server



Queue Configurations

Multi- Queues, Multi-servers



Kendall's Notation

(a/b/c) : (d/e/f)

where :

- **a** = arrival distribution;
  - M : Markovian (Poisson) arrival/or service,
  - E_k = Erlang service with shape parameter k
  - D : Constant service time,
  - G=General service time dist.
- **b** = service distribution
- **c** = number of servers
- **d** = queue discipline
  - FCFS (FIFO) : First Come First Served
  - SIRO : Service in Random Order
  - LCFS (LIFO) : Last Come First Served
  - GD : General Discipline
- **e** = maximum number allowed in the system (finite or infinite)
- **f** = size of calling source (finite or infinite)

Examples

(*M*/*M*/1) : (*FCFS*/∞/ ∞)
(*M*/*M*/1) : (*FCFS*/*N*/ ∞)
(*M*/*M*/*C*) : (*FCFS*/∞/ ∞)

Queue Characteristics

- Expected number of customers in the system (L_s)
- Expected number of customers in the queue (L_q)
- Expected waiting time in the system (W_s)
- Expected waiting time in queue (W_q)
- The server utilization factor (or busy period) (λ/μ) = ρ; also known as Traffic intensity or the clearing ratio

IMPORTANT CONSIDERATION

- The average service rate must always exceed the average arrival rate.
- Otherwise, the queue will grow to infinity.

$$\mu > \lambda$$

$$\infty$$

THERE WOULD BE NO SOLUTION !

$(M/M/1):(FCFS/\infty/\infty)$ Model Characteristics

Service is idle or there is no unit in the system

In the system, there can be 0 to ∞ events, that is

$$p_0 + p_1 + p_2 + p_3 + \ldots + p_n = 1$$

Since, $p_n = \left(\frac{\lambda}{\mu}\right)^n p_0$

$$p_0 + \left(\frac{\lambda}{\mu}\right)^1 p_0 + \left(\frac{\lambda}{\mu}\right)^2 p_0 + \left(\frac{\lambda}{\mu}\right)^3 p_0 + \ldots + \left(\frac{\lambda}{\mu}\right)^n p_0 = 1$$

$$p_0\left[\left(\frac{\lambda}{\mu}\right)^1 + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \ldots + \left(\frac{\lambda}{\mu}\right)^n\right] = 1 \Rightarrow p_0\left[\frac{1}{1-\frac{\lambda}{\mu}}\right] = 1$$

$$\therefore p_0 = \left(1 - \frac{\lambda}{\mu}\right)$$

$(M/M/1):(FCFS/\infty/\infty)$ Model Characteristics

Utilization factor or % of time for which service is busy

$$\rho = 1 - p_0 = \frac{\lambda}{\mu}$$

Probability of having no queue

$$p_0 + p_1 = \left(1-\frac{\lambda}{\mu}\right) + \frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right) = 1 - \left(\frac{\lambda}{\mu}\right)^2$$

Probability that queue is non-empty

$$1 - (p_0 + p_1) = \left(\frac{\lambda}{\mu}\right)^2$$

$(M/M/1):(FCFS/\infty/\infty)$ Model Characteristics

Expected number of units in the system

$$L_s = \sum_{n=0}^{\infty} n.p_n = 0.p_0 + 1.p_1 + 2.p_2 + 3.p_3 + \ldots + n.p_n$$

$$= 0 + 1.\left(\frac{\lambda}{\mu}\right)^1 p_0 + 2.\left(\frac{\lambda}{\mu}\right)^2 p_0 + 3.\left(\frac{\lambda}{\mu}\right)^3 p_0 + \ldots + n.\left(\frac{\lambda}{\mu}\right)^n p_0$$

$$= \frac{\lambda}{\mu} p_0\left[1 + 2.\left(\frac{\lambda}{\mu}\right)^1 + 3.\left(\frac{\lambda}{\mu}\right)^2 + \ldots + (n-1)\left(\frac{\lambda}{\mu}\right)^n\right]$$

$$= \frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right)\left[\frac{1}{\left(1-\frac{\lambda}{\mu}\right)^2}\right] = \frac{\lambda}{\mu - \lambda}$$

(M/M/1) : (FCFS/∞/ ∞) Model Characteristics

Expected number of units in the queue

$$L_q = 0.(p_0 + p_1) + 1.p_2 + 2.p_3 + \dots + (n-1).p_n$$

$$= 0 + 1.\left(\frac{\lambda}{\mu}\right)^2 p_0 + 2.\left(\frac{\lambda}{\mu}\right)^3 p_0 + 3.\left(\frac{\lambda}{\mu}\right)^3 p_0 + \dots + (n-1).\left(\frac{\lambda}{\mu}\right)^{n-1} p_0$$

$$= \left(\frac{\lambda}{\mu}\right)^2 p_0 \left[1 + 2.\left(\frac{\lambda}{\mu}\right)^1 + 3.\left(\frac{\lambda}{\mu}\right)^2 + \dots + (n-1)\left(\frac{\lambda}{\mu}\right)^n\right]$$

$$= \left(\frac{\lambda}{\mu}\right)^2 \left(1 - \frac{\lambda}{\mu}\right) \left[\frac{1}{\left(1 - \frac{\lambda}{\mu}\right)^2}\right] = \frac{\lambda}{\mu} . \frac{\lambda}{\mu - \lambda}$$

$$\therefore L_q = \rho . L_s$$

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(M/M/1) : (FCFS/∞/ ∞) Model Characteristics

Expected waiting time in the system

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu - \lambda}$$

Expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu} . \frac{1}{\mu - \lambda}$$

$$W_q = W_s - \frac{1}{\mu} = \frac{1}{\mu - \lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu} . \frac{1}{\mu - \lambda}$$

$$\therefore W_q = \rho . W_s$$

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(M/M/1) : (FCFS/∞/ ∞) Model Characteristics

Probability that the queue size is ≥ *k*

$$p(L_q \ge k) = \sum_{n=0}^{\infty} p_n - (p_0 + p_1 + \dots + p_{k-1})$$

$$= 1 - \left(p_0 + \frac{\lambda}{\mu} p_0 + \dots + \left(\frac{\lambda}{\mu}\right)^{k-1} p_0\right)$$

$$= 1 - p_0 \frac{1 - \left(\frac{\lambda}{\mu}\right)^k}{1 - \frac{\lambda}{\mu}} = \left(\frac{\lambda}{\mu}\right)^k$$

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(M/M/1) : (FCFS/∞/ ∞) Model Characteristics

Probability that the $W_q \ge t$

$$p(W_q \ge t) = \frac{\lambda}{\mu} e^{-(\mu - \lambda)t} \quad \textbf{and}$$

$$p(W_s \ge t) = e^{-(\mu - \lambda)t}$$

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(*M*/*M*/1) : (*FCFS*/∞/ ∞) Model Examples

A T.V. repairman finds that the time spent on his jobs has an exponential distribution with mean 30 minutes. If he repairs sets in the order in which they came in, and if the arrival of sets is approximately Poisson with an average rate of 10 per 8-hour day, what is repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?

In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time distribution is also exponential with an average 36 minutes, calculate the following:

(i) the mean queue size (line length), and
(ii) the probability that the queue size exceeds 10

If the input of trains increases to an average of 33 per day, what will be the change in (i) and (ii)?

(*M*/*M*/1) : (*FCFS*/∞/ ∞) Model Examples

The rate of arrival of customers at a public telephone booth follows Poisson distribution, with an average time of 10 minutes between one customer and the next. The duration of a phone call is assumed to follow exponential distribution, with mean time of 3 minutes.

(i) What is the probability that a person arriving at the booth will have to wait?
(ii) Estimate the fraction of a day that the phone will be in use.
(iii) What is the probability that it will take him more than 10 minutes altogether to wait for phone and complete his call?

(*M*/*M*/1) : (*FCFS*/∞/ ∞) Model Examples

A road transport company has one reservation clerk on duty at a time. He handles information of bus schedules and makes reservations. Customers arrive at a rate of 8 per hour and the clerk can service 12 customers on an average per hour. After stating your assumptions, answer the following:

i. What is the average number of customers waiting for the service of the clerk?
ii. What is the average time a customer has to wait before getting service?
iii. The management is contemplating to install a computer system to handle the information ;and reservations. This is expected to reduce the service time from 5 to 3 minutes. The additional cost of having the new system works out Rs.50 per day. If the cost of goodwill of having to wait is estimated to be 12 paise per minute spent waiting before being served. Should the company install the computer system? Assume 8 hours working day.

(*M*/*M*/1) : (*FCFS*/*N*/ ∞) Model Characteristics

In this system the maximum capacity of the system is restricted to *N*. Hence, maximum queue length is *N* – 1.

$$\rho = \frac{\lambda}{\mu}, \quad p_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$p_N = \rho^N p_0, \quad \lambda' = \lambda(1 - p_N)$$

$$L_s = \frac{\rho\left[1 - (1+N)\rho^N + N\rho^{N+1}\right]}{(1-\rho)(1-\rho^{N+1})}$$

$$L_q = L_s - 1 + p_0$$

$$W_s = \frac{L_s}{\lambda'} \quad \text{and} \quad W_q = \frac{L_q}{\lambda'}$$

(*M*/*M*/1) : (*FCFS*/*N*/ ∞) Model Example

Assume that the goods trains are coming in a yard at the rate of 30 trains per day and suppose that the inter-arrival times follow an exponential distribution. The service time for each train is assumed to be exponential with an average of 36 minutes. If the yard can admit 9 trains at a time, calculate the probability that the yard is empty and find the average queue length.

$$\lambda = \frac{1}{48} \text{ trains/min. and } \mu = \frac{1}{36} \text{ trains/min. and } \rho = \frac{\lambda}{\mu} = 0.75,$$

Prob. that yard is empty $p_0 = \dfrac{1-\rho}{1-\rho^{N+1}} = 0.265$

$$L_s = \frac{\rho\left[1-(1+N)\rho^N + N\rho^{N+1}\right]}{(1-\rho)\left(1-\rho^{N+1}\right)}$$

Average queue length $L_q = L_s - 1 + p_0 = 2.96$

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(*M*/*M*/*K*) : (*FCFS*/∞/ ∞) Model Characteristics

Number of service channels/servers = *k* and Utilization factor, $\rho = \lambda/k\mu$

Probability of having no units/customers in the system

$$p_0 = \frac{1}{\sum_{n=0}^{k-1} \frac{1}{n!} \cdot \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{k!} \cdot \left(\frac{\lambda}{\mu}\right)^k \cdot \frac{k\mu}{(k\mu - \lambda)}}$$

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(*M*/*M*/*K*) : (*FCFS*/∞/ ∞) Model Characteristics

Probability of having *n* units/customers in the system

$$p_n = \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} \cdot p_0 \qquad n < k$$

$$p_n = \frac{\left(\frac{\lambda}{\mu}\right)^n}{k!\,k^{n-k}} \cdot p_0 \qquad n \ge k$$

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(*M*/*M*/*K*) : (*FCFS*/∞/ ∞) Model Characteristics

Probability that a customer/unit has to wait

$$p(n \ge k) = \frac{\mu \cdot \left(\frac{\lambda}{\mu}\right)^k}{k!(k\mu - \lambda)} \cdot p_0$$

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(*M*/*M*/*K*) : (*FCFS*/∞/ ∞) Model Characteristics

Expected number of units in the system

$$L_s = \frac{\lambda\mu.\left(\frac{\lambda}{\mu}\right)^k}{(k-1)!(k\mu-\lambda)^2}.p_0 + \frac{\lambda}{\mu}$$

Expected number of units in the queue

$$L_q = L_s - \frac{\lambda}{\mu} = \frac{\lambda\mu.\left(\frac{\lambda}{\mu}\right)^k}{(k-1)!(k\mu-\lambda)^2}.p_0$$

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(*M*/*M*/*K*) : (*FCFS*/∞/ ∞) Model Characteristics

Average time an arrival spends in the system

$$W_s = \frac{L_s}{\lambda} = \frac{\mu.\left(\frac{\lambda}{\mu}\right)^k}{(k-1)!(k\mu-\lambda)^2}.p_0 + \frac{1}{\mu}$$

Expected number of units in the queue

$$W_q = W_s - \frac{1}{\mu} = \frac{L_q}{\lambda} = \frac{\mu.\left(\frac{\lambda}{\mu}\right)^k}{(k-1)!(k\mu-\lambda)^2}.p_0$$

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(*M*/*M*/*K*) : (*FCFS*/∞/ ∞) Model Example 1

A supermarket has two girls serving at the counters. The customers arrive in a Poisson fashion at the rate of 12 per hour. The service time for each customer is exponential with mean of 6 minutes. Find:

(i) the probability that an arriving customer has to wait for service,
(ii) the average number of customers in the system, and
(iii) the average time spent by a customer in the super-market.

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(*M*/*M*/*K*) : (*FCFS*/∞/ ∞) Model Example 2

A tax consulting firm has four stations in its office to receive people who have problems and complaints about their income, wealth and sale tax. Arrivals average 80 persons in an 8 hours service day. Each tax adviser spends an irregular amount of time servicing the arrivals which have been found to have an exponential distribution. The average service time is 20 minutes.

i. Calculate the average number of customers in the system.
ii. Average number of customers waiting to be serviced.
iii. Average time a customer spends in the system.
iv. Average waiting time for a customer.
v. How many hours each week does a tax adviser spend performing his job.
vi. What is the probability that a customer has to wait before he gets service.
vii. What is the expected number of idle tax advisers at any specified time.

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(*M*/*M*/*K*) : (*FCFS*/∞/ ∞) Model Example 2

Solution: λ = 10/hours, μ = 3/hours, k = 4, ρ = 0.833 and p_0 = 0.0213

i. Calculate the average number of customers in the system. (L_s) = 6.61

ii. Average number of customers waiting to be serviced. (L_q) = 3.28

iii. Average time a customer spends in the system. (W_s) = 0.66 hrs or 40 minutes

iv. Average waiting time for a customer. (W_q) = 0.33 hrs

v. How many hours each week does a tax adviser spend performing his job. (40xρ) = 33.3 hrs based on a 40 hrs week

vi. What is the probability that a customer has to wait before he gets service. $p(n \geq k)$ = 0.165

vii. What is the expected number of idle tax advisers at any specified time. $= 4p_0 + 3p_1 + 2p_2 + 1p_3 = 0.666$ So, less than one adviser is idle at any time.